Abstract

In this lecture we give a brief introduction to the field of Algorithmic Game Theory [2]. We define what games and solutions are and give a classification of games with respect to time, information and cooperation. Then, we define a game in strategic form and give some examples of strategic games.

1 Introduction

Game Theory [3, 1] provides mathematical models and tools to describe the behavior of agents that have to take decisions in situations where their actions are mutually influenced. The basic assumptions that underlie the theory are that decision-makers are rational, in the sense that each decision-maker pursue his objective and he can choose the best decision, and they reason strategically, in the sense that they take into account their knowledge or expectations of other decision-makers’ behaviors.

Example: Stag or hare Suppose that a set of \( n \) hunters set out to take a stag. They are fully aware that if they would all remain faithfully at their posts they will be able to take their prey and they will obtain \( 1/n \) of it. However, if a hare happens to pass near one of the hunters he can decide to leave his position (and thus having all hunters miss their part of the stag) and take his prey. Assume that each hunter prefers to take his part of the stag to taking an hare, but he prefers to take an hare to taking nothing. What has an hunter to do when he sees an hare? Notice that he has to consider the possibility that someone else sees the hare and gets it letting the stag to escape.

Game Theory started in 30’s as a branch of Mathematics with preliminary results on sum zero games (such as chess) but it was universally recognized as an important area with the fundamental work “Theory of Games and Economic Behavior” by Von Neumann-Morgenstern. In the following years it rapidly gained interest in wide fields of social sciences as a tool to understand the behaviors of interacting decision-makers. Nowadays Game Theory is a fundamental tool in Economics (competition among economic agents on the market can be seen as a game where each agent has to fix prices and productions to maximize its utilities), Social Sciences (politicians decide their strategies in order to gain consensus), Biology (living beings are in a perpetual competition with other beings to survive and gain domain), Engineering (transportation planning, logistics).

Recently, Game Theory is getting more and more important also in the field of Computer Science. The end of the 20th century was characterized by the world-wide diffusion of the Internet and the emergence of the Web that changed fundamentally social relationships. As Tim Roughgarden notices in [4], “this revolution changed the social role of computers, which evolved from a stand-alone well-understood machine for executing software to a conduit for global communication,
content-dissemination and commerce”. As a consequence of this revolution computer scientists started to formulate novel problems and design novel techniques that can help in designing and analyzing applications and services which have to work on modern networks that are essentially decentralized and uncoordinated.

In this case, the traditional model of distributed computing where a centrally planned algorithm is distributed among nodes of the network prescribing how each node has to compute and cooperate with other nodes, is of no help. In fact, modern networks are characterized by a great part of uncoordination: they are formed by heterogeneous nodes, maybe belonging to different organizations with different interests, not controlled by any authority, that interact in a constantly changing context. In this scenario, the network is intrinsically a common playground for a large number of users with different degrees of collaboration/competition, that can dynamically change with respect to their own contingent objectives to pursue. Thus, a new point of view has to be adopted in studying how these networks work. In fact, the global behaviour of the system will be the result of actions autonomously decided by selfish agents that control network resources and whose behaviour is driven by the pursuing of personal utilities. The objective of the designers is to have these nodes to cooperate and accomplish tasks that are difficult or even beyond the capabilities of the nodes.

At the end of the century it became clear to researchers that Game Theory for its capacity to describe the interactions between decision-makers was the most promising tool for supporting this new point of view. In last ten years we assisted to an extraordinary proliferacy of studies at the interface of theoretical computer science, game theory and microeconomics. Researchers coming from different disciplines fruitfully interacted showing an extraordinary example of interdisciplinary research that brought to a mutual enrichment of the original disciplines and gave rise to a new field that has been named as Algorithmic Game Theory (AGT).

The central research themes in AGT differ from those in classical microeconomics and game theory in important, albeit predictable, respects. Firstly in application areas: Internet-like networks and non-traditional auctions (such as digital goods and search auctions) motivate much of the work in AGT. Secondly in its quantitative engineering approach: AGT research typically models applications via concrete optimization problems and seeks optimal solutions, impossibility results, upper and lower bounds on feasible approximation guarantees, and so on. Finally, AGT usually adopts reasonable (e.g., polynomial-time) computational complexity as a binding constraint on the feasible behavior of system designers and participants. These themes, which have played only a peripheral role in traditional game theory, give AGT its distinct character and relevance. AGT also connects to traditional theoretical computer science in remarkably diverse and rich ways. For example, recent work on auction design has been informed by techniques ranging from primal-dual algorithms to communication complexity; quantitative analyses of game-theoretic equilibria have drawn on tools familiar from approximation algorithm design, programming and potential function arguments; and studying the complexity of computing such equilibria has resurrected interest in previously obscure complexity classes that were originally motivated by local search and combinatorial topology problems.

2 Games and solutions

A game is a description of any situation where several agents have to take decisions that are mutually related. Notice that a game is only a formal description of the strategic interactions between decision-makers’ behaviors, including the constraints on the actions that players can take and their interests. The game does not specify which actions players do take. A solution,
instead, is a systematic description of the outcomes that may emerge in a family of games. Game Theory deals with the description of reasonable solutions for classes of games and examines properties of these solutions.

As stated in the introduction, the game theoretical models assume that players are rational and thus they are aware of their alternatives, form expectations about any unknown, have clear preferences and choose their actions according to some process of optimization. In the simpler case, where no uncertainty exists, our model of rational choice includes:

- A set $A$ of actions from which the decision-maker makes a choice;
- A set $C$ of possible consequences of their actions (also defined outcomes of the game);
- A consequence function $g : A \rightarrow C$, that associates outcomes to actions;
- for each agent $i$, a preference relation $\succeq_i$ on the set $C$.

Sometimes, it may be useful to describe the preferences with a utility function $u_i : C \rightarrow \mathbb{R}$, that associates to each outcome a real value that measures how much the agent likes that outcome. In this case, we have that

$$\forall x, y \in C \quad x \succeq_i y \iff u_i(x) \geq u_i(y).$$

A rational agent is able to select the subset of actions $B \subseteq A$ that are feasible to him and to compute the action $x^* \in B$ that is his best choice, that is

$$u_i(x^*) = \max_{x \in B} \{u_i(g(x))\}.$$ 

Game Theory does not consider different abilities of agents in valuating the game. We assume that all players have same computational capacities and they are able to select their best strategy. Reality is different and ability of the players is a fundamental issue (consider how boring chess would be if all players had same strategic capacities).

In a more realistic model each agent has to take decisions in a situation of uncertainty (this uncertainty can regard their strategies, state of the game, behavior of the other players, the consequences of their actions, etc.), In these cases each rational player tries to maximize his expected utility.

### 2.1 Classification of games

We can classify games with respect to three parameters: cooperation, information and time.

**cooperation** we distinguish between **non-cooperative games** where every agent plays for maximizing his own utility based on his belief of the other agents’ behaviour and **cooperative games** where agents cooperate in deciding their behaviours in order to maximize some sort of global utility;

**information** we distinguish between **perfect (full) information games** where agents have complete information of the game (that is the agent knows other agents’ moves, and he knows that they know and he knows that they know that other know, and so on) and **imperfect (partial) information games** where agents have only a partial knowledge of the game;
time we distinguish between games in strategic (normal) form where each player has to decide before the playing his strategy (that can consist of several moves) with no interaction with other players and games in extensive form that are played in steps, where at each step a player takes a decision based on his knowledge of the status of the game and of the history of the game.

Most of the games considered in this course are non-cooperative, strategic and with full information.

3 Strategic games

A strategic game is a model of interactive decision-making where each player chooses his plan of action once and for all and all players make their choices at the same time. You can think of this game as a one-shot game that is played once and consists of a single move. On the other hand, games in phases can also be described as strategic games. In this case the player has to decide his full strategy, consisting of his actions in any possible state of the game.

Formally, a game in strategic (normal) form is defined as a triple $(N, (A_i), (\succeq_i))$, where:

- $N$ is finite set of players,
- for each player $i \in N$, $A_i$ is the nonempty set of the actions available to player $i$,
- for each player $i \in N$, $\succeq_i$ is the preference relation of player $i$ on the set $(A_1 \times A_2 \times \cdots \times A_n)$.

If the sets of actions of all players are finite we say that the game is finite.

For sake of simplicity we will define a set $O$ of the possible outcomes of the game, a function $g : (A_1 \times A_2 \times \cdots \times A_n) \rightarrow O$ and for each player $i$ a utility function $u_i : O \rightarrow \mathbb{R}$ such that

$$\forall x, y \in O \quad x \succeq_i y \quad iff \quad u_i(x) \geq u_i(y).$$

Thus the strategic game is defined as $(N, (A_i), (u_i))$. In this game, a rational player that knows that other players play actions $a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ is able to select the action $a_i$ such that $(a_i, a_{-i}) = \arg\max_{a_i \in A} u_i(g(a))$.

When there are only two players it is possible to describe succinctly the game as a matrix where the row entries $(a_1, a_2, \cdots, a_n)$ are the actions available for player 1, the column entries $(b_1, b_2, \cdots, b_n)$ are actions available for player 2 and each entry $(i, j)$ contains the utility associated by the two players to the outcome associated to the action profile $(a_i, b_j)$.

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<th>$b_2$</th>
<th>$b_3$</th>
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We define a solution of the game as the profile of actions taken by the players in a realization of the game. Thus, solutions are elements taken from the set $(A_1 \times A_2 \times \cdots \times A_n)$. Most of the time we are interested in solutions that have specific properties. For example, we could be interested in “stable solutions”, where each player, observed actions of other players, is not interested in changing his strategy.
Example 1: Prisoner’s Dilemma  Two suspects in a crime are put in two separate cells. If they both confess, each will be sentenced to four years in prison. If only one confess, he will be freed and used as a witness against the other, who will receive a sentence of five years. If neither confess, they both will be sentenced for a minor offense to two years in prison. Each player prefers to spend as few years in prison as possible. This situation can be described as a strategic game with two players, where each player has two possible actions: Confess or Not Confess. The game is completely described by the following matrix, sometimes called payoff matrix:

<table>
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<th>C</th>
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<tbody>
<tr>
<td>C</td>
<td>4, 4</td>
<td>0, 5</td>
</tr>
<tr>
<td>N</td>
<td>5, 0</td>
<td>2, 2</td>
</tr>
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At a first glance it may appear that the best strategy of the suspects for playing this game is to agree in advance not to confess (and so be sentenced for two years). However, the hypothesis that suspects are rational and play for spending as few years in prison as possible convince each suspect that when the game is played his colleague will not respect the agreement (and so does he). We observe that in this game there is only a solution that is stable and it is \((C, C)\).

Notice that the formulation of the game is no way bounded to a specific context but can be applied to a wide range of situations where two players have two strategies and their preferences are structured as in the previous payoff matrix. Next example is taken from a completely different context (routing traffic over Internet) but it will result in the same payoff matrix and thus it is possible to analyze the behaviours of the players in the same way as for the suspects of the previous example.

Example 2: ISP Routing Game  Consider two ISP service providers that need to send traffic to each other. Assume that each provider has his own network and these networks are connected as in figure.

![Figure 1: The ISP routing game](image)

The two networks are connected through two peering points, called \(C\) and \(N\). Thus if ISP 1
wants to send traffic from $s_1$ to $t_1$ it has to choose to send it through $C$ or $N$. Usually, ISPs are selfish and try to send traffic along the shortest (or least cost) path. However, in this case when an ISP selects a route for the traffic originated in his network and directed to the other network is implicitly influencing the loads on links of the other ISP. Thus, in our example if ISP 1 can choose to send traffic from $s_1$ to $t_1$ through the closest node $C$ paying 1 but in this case his colleague should pay 5 or to send the traffic through the farthest node $N$ paying 2 and having his colleague paying the same cost. ISP 2 has the same strategic alternatives for routing traffic from $s_2$ to $t_2$.

It can be easily seen that in this example there are two players with two alternative strategies and the payoff matrix that describes the game is equal to the matrix of the Prisoner’s Dilemma.

References


